

On invariants of link maps in dimension four

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Abstract

We affirmatively address the question of whether the proposed link homotopy invariant ω of Li is well-defined. It is also shown that if one wishes to adapt the homotopy invariant τ of Schneiderman-Teichner to a link homotopy invariant of link maps, the result coincides with ω .

1 Introduction

A link map $S^2 \cup S^2 \rightarrow S^4$ is a map from a union of 2-spheres with pairwise disjoint images, and a link homotopy is a homotopy through link maps. To a link map f , Kirk ([4], [5]) assigned a pair of integer polynomials $\sigma(f) = (\sigma_+(f), \sigma_-(f))$ which is invariant under link homotopy and vanishes if f is link homotopic to a link map that embeds either component. He posed the still-open problem of whether $\sigma(f) = (0, 0)$ is sufficient to link nullhomotope f . In [6], Li sought to define a link homotopy invariant $\omega(f) = (\omega_+(f), \omega_-(f))$ to detect link maps in the kernel of σ . When $\sigma_\pm(f) = 0$, the mod 2 integer $\omega_\pm(f)$ obstructs embedding by counting (after a link homotopy) weighted intersections between $f(S_\pm^2)$ and its Whitney disks in the complement of $f(S_\mp^2)$. While the examples with $\sigma(f) = (0, 0)$ but $\omega(f) \neq (0, 0)$ in that paper were found to be in error by Pilz ([8]), the latter did not address another issue. Namely, the proof that ω is invariant under link homotopy relies implicitly on the assumption that a pair of link homotopic abelian link maps are link homotopic through *abelian* link maps. The first purpose of this note is to prove this assumption correct and that ω is a link homotopy invariant.

Theorem 1. *If f and g are link homotopic link maps such that $\sigma(f) = (0, 0) = \sigma(g)$, then $\omega(f) = \omega(g)$.*

The invariants σ and ω may then be viewed, respectively, as primary and secondary obstructions to link homotoping to an embedding. For the problem of homotoping a map $S^2 \rightarrow Y^4$ to an embedding, the homotopy invariants μ of Wall ([10]) and τ of Schneiderman and Teichner ([9]) form an analogous pair of obstructions. Our second purpose is to show that if one adapts τ to the setting of link homotopy in the natural way, one obtains ω . For a link map $f : S_+^2 \cup S_-^2 \rightarrow S^4$, where the signs are used to distinguish each component, write $X_\pm = S^4 \setminus f(S_\mp^2)$ and let f_\pm denote the restricted map $f|_{S_\pm^2} : S_\pm^2 \rightarrow X_\mp$.

Theorem 2. *Let f be a link map with $\sigma_+(f) = 0$. Then f is link homotopic to a link map g such that $\tau(g_+)$ is defined, and one has that $\tau(g_+) = 0$ if and only if $\omega_+(f) = 0$.*

Assume all manifolds are equipped with basepoints and orientations arbitrarily unless otherwise specified.

2 Proof of Theorem 1

A link map f is said to be *abelian* if $\pi_1(X_+) \cong \mathbb{Z}$ and $\pi_1(X_-) \cong \mathbb{Z}$, and an abelian link homotopy is a link homotopy through abelian link maps. We will say f is *good* if it is abelian and each restricted map f_{\pm} is a self-transverse immersion with vanishing signed self-intersection number. Theorem 1 will be shown using the following lemma.

Lemma 1. *If f and g are regularly homotopic good link maps such that $\sigma(f) = (0, 0) = \sigma(g)$, then $\omega(f) = \omega(g)$.*

Proof of Theorem 1. A link map f may be first be perturbed so that it restricts to a self-transverse immersion on each component 2-sphere; local cusp homotopies may then be performed so that these immersions each have vanishing signed self-intersection number. Finger moves of $f(S_+^2)$ in the complement of $f(S_-^2)$, followed by finger moves of $f(S_-^2)$ in the complement of $f(S_+^2)$, may then serve to abelianize $\pi_1(X_+)$ and $\pi_1(X_-)$ (see [1, p. 205]; also [4], [6]). Denote the resulting good link map by f' . In [6], the author gives an algorithm for computing the pair of mod 2 integers $\omega(f') = (\omega_+(f'), \omega_-(f'))$ and defines $\omega(f) = \omega(f')$. Suppose f'' is another good link homotopy representative of f . Then f' and f'' are regularly homotopic by [5, Theorem 2.4], so $\omega(f'') = \omega(f')$ by Lemma 1. Thus ω does not depend on the choice of good representative. Now, if g is link homotopic to f , and g' is a good link homotopy representative of g , the same argument shows that $\omega(g') = \omega(f')$, so (as defined) $\omega(f) = \omega(g)$. \square

It remains to prove Lemma 1. We make use of ideas from [3], to which the reader is referred for more details on finger and Whitney moves along chords. Throughout the rest of this paper, Y will denote a 4-manifold. Let $k : S^2 \rightarrow Y$ be an immersion. A *chord* γ attached to $k(S^2)$ is a continuous arc in Y whose endpoints are distinct points of $k(S^2)$ (minus its double points) and whose interior is disjoint from $k(S^2)$. A chord is *simple* if this arc is an embedding. If two simple chords γ and γ' for $k(S^2)$ are ambient isotopic in Y by an isotopy fixing $k(S^2)$, then a finger move along either chord yields an ambient isotopic immersion. The following result is a ready consequence of transversality and the isotopy extension theorem.

Lemma 2. *Let C be a compact subset of a 4-manifold Y , and let $k : S^2 \rightarrow Y$ be an immersion. Suppose α and β are simple chords on $k(S^2)$ with common endpoints p, q . If α and β are path homotopic in $Y \setminus C$ through chords on $k(S^2)$, then α and β are ambient isotopic in Y by an isotopy that carries α to β and fixes $k(S^2)$ and C .*

We first show that, roughly speaking, finger moves and Whitney moves of a single component of a link map in the complement of the other component commute.

Lemma 3. *Let f be a link map such that the restricted maps f_{\pm} are self-transverse immersions. Suppose that an immersion g_+ is obtained from f_+ by*

performing a Whitney move, followed by a finger move, in $S^4 \setminus f(S_-^2)$. Then, up to ambient isotopy in S^4 fixing $f(S_-^2)$, g_+ may be obtained from f_+ by performing a finger move, followed by a Whitney move, in $S^4 \setminus f(S_-^2)$.

Proof. By the hypotheses, there is an intermediate link map f' such that $f'_- = f_-$ and a Whitney move performed in a 4-ball $B \subset S^4 \setminus f(S_-^2)$ changes f_+ to f'_+ , and a finger move of f'_+ along a chord $\gamma \subset S^4 \setminus f(S_-^2)$ changes f'_+ to g_+ .

If γ is disjoint from B , then the lemma is immediate. Otherwise, we may assume that γ intersects $B \setminus f'(S_+^2)$ along the interior of γ in a collection of $n \geq 0$ properly embedded arcs. One may then homotop γ in a collar of $\partial B \setminus f'(S_+^2)$ to a union $\hat{\gamma} \cup_{i=1}^n \alpha_i$, of a simple chord $\hat{\gamma} \subset S^4 \setminus f(S_-^2)$ on $f'(S_+^2)$ that intersects B at precisely one point $p \in \partial B$, and n simple loops $\{\alpha_i\}_{i=1}^n$ in $B \setminus f'(S_+^2)$ based at p . But as inclusion induces a surjection $\pi_1(\partial B \setminus f'(S_+^2), p) \rightarrow \pi_1(B \setminus f'(S_+^2), p)$, we may further deduce that γ is path homotopic in $S^4 \setminus f(S_-^2)$ through chords on $f'(S_+^2)$ to a simple chord γ' that misses B . Thus, by Lemma 2 there is an ambient isotopy in S^4 from γ to γ' that fixes $f'(S_+^2)$ and $f(S_-^2)$. \square

We further require that, roughly speaking, a Whitney move of one component of a link map commutes with a finger move of the other. The proof is similar to that of Lemma 3 but we include it for completeness.

Lemma 4. *Suppose that f and g are good link maps such that g_+ is obtained from f_+ by performing a Whitney move in $S^4 \setminus f(S_-^2)$ and g_- is obtained from f_- by performing a finger move in $S^4 \setminus g(S_+^2)$. Then (up to ambient isotopy in S^4 fixing $f(S_+^2)$) g_- may be obtained from f_- by performing a finger move in $S^4 \setminus f(S_+^2)$ and (up to ambient isotopy in S^4 fixing $g(S_-^2)$) g_+ may be obtained by performing a Whitney move of f_+ in $S^4 \setminus g(S_-^2)$.*

Proof. Let B be a 4-ball in $S^4 \setminus f(S_-^2)$ such that a Whitney move performed in B changes f_+ to g_+ , and let γ be a simple chord in $S^4 \setminus g(S_+^2)$ on $f(S_-^2)$ such that a finger move of f_- along γ changes f_- to g_- . If γ is disjoint from B then the lemma holds without the need for an additional isotopy (note that $f(S_+^2)$ and $g(S_+^2)$ coincide outside B). Otherwise, we may assume that γ intersects $B \setminus g(S_+^2)$ along the interior of γ in a finite collection of properly embedded arcs. Since inclusion induces a surjection $\pi_1(\partial B \setminus g(S_+^2)) \rightarrow \pi_1(B \setminus g(S_+^2))$, as in the proof of Lemma 3 one may path homotop γ through chords on $g(S_+^2)$ to a simple chord that misses B . Now apply Lemma 2. \square

We can now prove that if two good link maps are regularly link homotopic then they are connected by an *abelian* link homotopy.

Lemma 5. *If f and g are regularly homotopic good link maps, then there is a regular homotopy from f to g consisting of a sequence of abelian link homotopies that alternately fix one component.*

Proof. As in the proof of [5, Theorem 2.4], there is a regular homotopy taking f to g consisting of a sequence of regular homotopies that alternately fix one component. By Lemmas 3 and 4 this sequence can be chosen to first consist of

finger moves (and ambient isotopies) alternately fixing one component, carrying f to a link map f' , then a sequence of Whitney moves (and ambient isotopies) alternately fixing one component, carrying f' to g .

Now, finger moves and ambient isotopy preserve abelianess, so f and f' are connected by a sequence of abelian, regular link homotopies alternately fixing one component. On the other hand, f' is obtained from g by a sequence of finger moves (and ambient isotopies) alternatively fixing one component, so these link maps are also connected by abelian, regular link homotopies alternately fixing one component. \square

Proof of Lemma 1. By Lemma 5, the link map f is carried to g by a sequence of abelian, regular link homotopies that alternately fix each component. Lemma 1 then follows from the following proposition, which can be deduced from the proof of [6, Proposition 4.2] (and which is expounded upon in [8, Satz 4.14]). \square

Proposition 1. *Let h be a good link map with $\sigma_+(h) = 0$.*

- (i) *If a good link map h' is obtained from h by performing a regular homotopy of h_+ in $S^4 \setminus h(S_-^2)$, then $\omega_+(h') = \omega_+(h)$. \square*
- (ii) *If a good link map h' is obtained from h by performing a regular homotopy of h_- in $S^4 \setminus h(S_+^2)$, then $\omega_+(h') = \omega_+(h)$.*

Remark. Part (i) of this proposition is essentially a special case of the proof in [9] that the τ -invariant is well-defined, while part (ii) is unique in that the ambient manifold X_- , into which h_+ maps, is allowed to change.

3 Proof of Theorem 2

We begin with some preliminary definitions concerning algebraic intersections of immersed surfaces in 4-manifolds. The reader is referred to [2] for more details on the subject.

3.1 Intersection numbers in 4-manifolds

Suppose A and B are properly immersed, self-transverse 2-spheres or 2-disks in a 4-manifold Y . Suppose further that A and B are transverse and that each is equipped with a path (a *whisker*) connecting it to the basepoint of Y .

For an intersection point $x \in A \cap B$, let $\lambda(A, B)_x \in \pi_1(Y)$ denote the homotopy class of a loop that runs from the basepoint of Y to A along its whisker, then along A to x , and back to the basepoint along B and its whisker. Define $\text{sign}_{A,B}(x)$ to be 1 or -1 depending on whether or not, respectively, the orientations of A and B induce the orientation of Y at x . The (algebraic) intersection “number” $\lambda(A, B)$ between A and B is then defined as the sum in the group ring $\mathbb{Z}[\pi_1(Y)]$ of $\text{sign}(x)\lambda(A, B)_x$ over all such intersection points. The value of $\lambda(A, B)$ is invariant under homotopy rel boundary of A or B ([2]), but depends on the choice of basepoint of Y and the choices of whiskers and orientations.

The following two observations will be useful. If $x, y \in A \cap B$, then the product of $\pi_1(Y)$ -elements $\lambda(A, B)_x (\lambda(A, B)_y)^{-1}$ is represented by a loop that runs from the basepoint to A along its whisker, along A to x , then along B to y , and back to the basepoint along A and its whisker. Secondly, if $D_A \subset A$ is a 2-disk that is equipped with the same whisker and oriented consistently with A , then $\lambda(A, B)_x = \lambda(D_A, B)_x$ and $\text{sign}_{A, B}(x) = \text{sign}_{D_A, B}(x)$ for each $x \in D_A \cap B$.

3.1.1 Surgering tori to 2-spheres

Suppose T is an embedded torus (or punctured torus, resp.) in $Y \setminus \text{int } B$ and suppose there is a circle $\delta_1 \subset T$ that is nullhomotopic in Y . Choose an immersed 2-disk D in Y that is bound by δ_1 and transverse to T and B , and choose a normal vector field ϕ to δ_1 on T . Let $\delta'_1 \subset T$ denote a nearby push-off of δ_1 along ϕ . Extend ϕ over D and let D' denote a pushoff of D along ϕ , bound by δ'_1 and which we may assume is also transverse to B . If D is oriented and D' has the orientation induced as the pushoff, then intersections between B and $D \cup D'$ occur as finitely many nearby pairs of points $\{x_i, x'_i\}_{i=1}^n$, where $x_i \in \text{int } D$ and $x'_i \in \text{int } D'$ are of opposite sign. Thus, removing from T the interior of the annulus bound by $\delta_1 \cup \delta'_1$ and attaching $D \cup D'$ yields an immersed 2-sphere (or 2-disk with boundary ∂T , resp.) S in Y such that the intersections between B and S are transverse and occur precisely at the pairs of points $\{x_i, x'_i\}_{i=1}^n$. Furthermore, the algebraic intersections between S and B may be calculated using the following lemma. Let $[\alpha]$ denote the class in $\pi_1(Y)$ of a based loop α in Y , let $\bar{\gamma}$ denote the reverse of a path γ , and let $*$ denote composition of paths.

Lemma 6. *Let δ_2 be an oriented, simple circle on T that intersects each of δ_1 and δ'_1 exactly once, at points z and z' (respectively), and is tangent to ϕ at z . Let ι be a path in Y from its basepoint to z . If S and D are oriented consistently and both equipped with the whisker ι , then*

$$\lambda(S, B) = (1 - [\iota * \delta_2 * \bar{\iota}])\lambda(D, B).$$

Proof. For each i , let γ_i be a path on D connecting z to x_i (that does not pass through any double points) and let γ'_i be its pushoff along ϕ , connecting z' to x'_i . Let β_i be a path on B from x_i to x'_i (that does not pass through any double points), and let $\widehat{\delta}_2$ be the arc $\delta_2 \cap S$, oriented to run from z' to z . Then the product $\lambda(S, B)_{x_i} (\lambda(S, B)_{x'_i})^{-1}$ is represented by the loop

$$\iota * \gamma_i * \beta_i * \overline{\gamma'_i} * \widehat{\delta}_2 * \bar{\iota}. \quad (1)$$

Homotoping S (rel boundary) by collapsing D' onto D except near its intersections with B , one sees that the loop (1) is homotopic in Y to the loop $\iota * \delta_2 * \bar{\iota}$. Thus, equipping D with the whisker ι and the same orientation as S , we have

$$\lambda(S, B)_{x'_i} = [\iota * \bar{\delta}_2 * \bar{\iota}] \lambda(S, B)_{x_i} = [\iota * \bar{\delta}_2 * \bar{\iota}] \lambda(D, B)_{x_i}$$

and $\text{sign}_{S,B}(x'_i) = \text{sign}_{D',B}(x'_i) = -\text{sign}_{D,B}(x_i)$. Summing over all such pairs of intersections yields

$$\begin{aligned}\lambda(S, B) &= \sum_i \text{sign}_{S,B}(x_i) \lambda(S, B)_{x_i} + \text{sign}_{S,B}(x'_i) \lambda(S, B)_{x'_i} \\ &= \sum_i (1 - [\iota * \bar{\delta}_2 * \bar{\iota}]) \text{sign}_{D,B}(x_i) \lambda(D, B)_{x_i} \\ &= (1 - [\iota * \bar{\delta}_2 * \bar{\iota}]) \lambda(D, B). \quad \square\end{aligned}$$

3.2 Unknotted immersions and link maps

Two immersions $k_0, k_1 : S^2 \rightarrow \mathbb{R}^4$ are said to be *equivalent* if there are orientation-preserving self-diffeomorphisms h of S^2 and H of \mathbb{R}^4 , respectively, such that $k_1 \circ h = H \circ k_0$. Denote the standard embedding $S^2 \subset \mathbb{R}^4$ by u_0^0 . By applying local cusp homotopies, d of positive sign and e of negative sign, to u_0^0 , one obtains an *unknotted* immersion, denoted $u_d^e : S^2 \rightarrow \mathbb{R}^4$. Note that u_d^e is unique up to equivalence; we say that an immersion $k : S^2 \rightarrow \mathbb{R}^4$ (or its image) is unknotted if k is equivalent to u_d^e for some $d, e \geq 0$. See [3] for more details.

Identify $S^4 = \mathbb{R}^4 \cup \{\infty\}$.

Lemma 7. *A link map f is link homotopic to a good link map g such that $g(S_-^2)$ is unknotted in $\mathbb{R}^4 \subset S^4$.*

Proof. As in the proof of Theorem 1, we may assume after a link homotopy that f is a good link map. By [3, Lemma 3], there is a family of disjoint chords attached to $f(S_-^2)$ such that finger moves along them change $f(S_-^2)$ into an unknotted immersion in $\mathbb{R}^4 = S^4 \setminus \{\infty\}$. As these chords may be assumed to miss $f(S_+^2)$, we have the required result. \square

For an immersed 2-sphere A in a 4-manifold Y , let $\omega_2(A) \in \mathbb{Z}_2$ denote the second Stiefel Whitney number of the normal bundle of A in Y . The results in [7] readily generalize to give the following.

Lemma 8. *Suppose $k : S^2 \rightarrow \mathbb{R}^4$ is an unknotted, self-transverse immersion with d double points, and let Y denote the complement in S^4 of $k(S^2)$. Then $\pi_2(X_-)$ is a free $\mathbb{Z}[\mathbb{Z}]$ -module on d generators, the Hurewicz map $\pi_2(Y) \rightarrow H_2(Y)$ surjects and $\omega_2(A) = 0$ for any immersed 2-sphere A in Y .*

Remark. Indeed, the complement of an open tubular neighborhood of $k(S^2)$ in S^4 has a handlebody decomposition consisting of one 0-handle, one 1-handle, and d zero-framed 2-handles attached along unknotted circles in S^3 which are nullhomotopic in the boundary of the union of the 0- and 1-handle.

Let $k : S^2 \rightarrow Y$ be a self-transverse immersion and suppose p is a double point of $k(S^2)$. An *accessory circle* for p is an (oriented) simple circle on $k(S^2)$ that passes through exactly one double point, p , and changes sheets there.

Lemma 9. *Let f be a good link map such that $f(S_-^2)$ is unknotted in $\mathbb{R}^4 \subset S^4$. Equip $f(S_+^2)$ with a whisker in X_- and fix an identification of $\pi_1(X_-)$ with $\mathbb{Z}\langle s \rangle$ so as to write $\mathbb{Z}[\pi_1(X_-)] = \mathbb{Z}[s, s^{-1}]$. Label the double points of $f(S_-^2)$ by $\{p_i\}_{i=1}^d$ and choose an accessory circle α_i for p_i for each $1 \leq i \leq d$. Then $\pi_2(X_-) \cong (\bigoplus_{i=1}^d \mathbb{Z})[s, s^{-1}]$ and there is a $\mathbb{Z}[s, s^{-1}]$ -basis represented by self-transverse, immersed, whiskered 2-spheres $\{A_i\}_{i=1}^d$ in X_- with the following properties. For each $1 \leq i \leq d$, there is an integer Laurent polynomial $q_i \in \mathbb{Z}[s, s^{-1}]$ such that*

$$\lambda(f(S_+^2), A_i) = (1 - s)^2 q_i(s)$$

and $q_i(1) = \text{lk}(f(S_+^2), \alpha_i)$. Moreover, if for any $1 \leq j \leq d$ the loop α_j bounds a 2-disk in S^4 that intersects $f(S_+^2)$ exactly once, then we may choose A_j so that

$$\lambda(f(S_+^2), A_j) = (1 - s)^2.$$

Proof. For $t_0, t'_0 \in \mathbb{R}$, $t'_0 > t_0$, let $\mathbb{R}^3[t_0]$ denote the hyperplane of \mathbb{R}^4 whose fourth coordinate t is t_0 , and let $\mathbb{R}^3[t_0, t'_0] = \{(x, t) \in \mathbb{R}^4 : x \in \mathbb{R}^3, t_0 \leq t \leq t'_0\}$. Figure 1 gives a “moving picture” description of an immersed 2-disk U (appearing as an arc in each slice $\mathbb{R}^3[t_0]$, $t_0 \in [-1, 1]$) in a 4-ball $N \subset \mathbb{R}^3[-1, 1]$, with a single self-transverse double point $p \in \mathbb{R}^3[0]$. In this figure we have labeled a loop $\alpha \subset \mathbb{R}^3[0]$ on U that changes sheets at p and bounds a 2-disk D . For each $1 \leq i \leq d$, let \widehat{U}_i be a 2-disk on S_-^2 that contains the two preimages of the double point p_i , and no other double point preimages. There is a diffeomorphism Γ_i of N onto a 4-ball neighborhood of p_i in S^4 that takes U to $f(\widehat{U}_i)$, α to α_i and p to p_i . Choose a 4-ball neighborhood $N^+ \subset N$ of p so that (the smaller 4-ball) $\Gamma_i(N^+)$ is disjoint from $f(S_+^2)$. There is a torus T in $N^+ \setminus U$ that

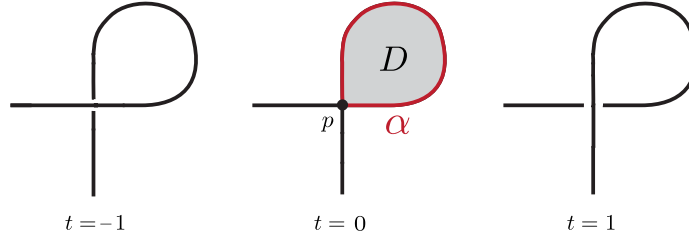


Figure 1

intersects D exactly once; see Figure 2. The torus appears as a cylinder in each of $\mathbb{R}^3[-1]$ and $\mathbb{R}^3[1]$, and appears as a pair of circles in $\mathbb{R}^3[t_0]$ for $t_0 \in (-1, 1)$. By Alexander duality the linking pairing

$$H_2(X_-) \times H_1(f(S_-^2)) \rightarrow \mathbb{Z}$$

defined by $(R, v) \mapsto R \cdot \Upsilon$, where $v = \partial \Upsilon \subset S^4$, is nondegenerate. Thus, as the loops $\{\alpha_i\}_i$ represent a basis for $H_1(f(S_-^2)) \cong \mathbb{Z}^d$, we have that $H_2(X_-) \cong \mathbb{Z}^d$ and (after orienting) the so-called *linking tori* $\{T_i\}_i$, defined by $T_i = \Gamma_i(T)$,

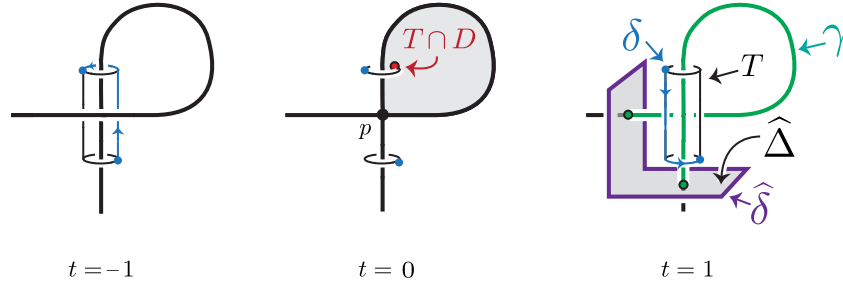


Figure 2

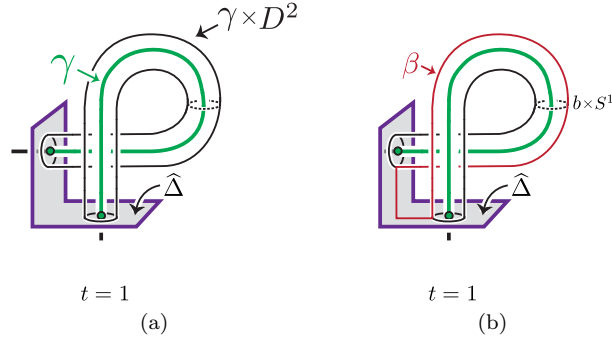


Figure 3

represent a basis. We proceed to apply the construction of (3.1.1) (twice, successively) to turn these tori into 2-spheres.

In Figure 2 we have illustrated an oriented circle δ on T which intersects $\mathbb{R}^3[-1]$ and $\mathbb{R}^3[1]$ each in an arc, and appears as a pair of points in $\mathbb{R}^3[t_0]$ for $t_0 \in (-1, 1)$. Notice that δ is isotopic in $N^+ \setminus U$ to the circle $\hat{\delta} \subset \mathbb{R}^3[1]$ that is also illustrated in Figure 2. By attaching the trace of such an isotopy to the 2-disk $\hat{\Delta} \subset \mathbb{R}^3[1]$ illustrated, bound by $\hat{\delta}$, one may obtain an embedded 2-disk $\Delta \subset N^+$ that is bound by δ and intersects U precisely where $\hat{\Delta}$ does. These intersection points are the endpoints of an arc $\gamma \subset \mathbb{R}^3[1]$ on U , shown in Figure 2. Let $\gamma_i = \Gamma_i(\gamma) \subset f(S_-^2)$. In Figure 3(a) we have illustrated in $\mathbb{R}^3[1]$ the restriction of a tubular neighborhood of U to γ . Identifying this tubular neighborhood with $\gamma \times D^2$, we may assume the embedding Γ_i carries $\gamma \times D^2$ onto the restriction over γ_i of a tubular neighborhood of $f(S_-^2)$ that is disjoint from $f(S_+^2)$. Let Θ be the punctured torus in $N \setminus U$ given by

$$\Theta = \Delta \setminus (\partial\gamma \times \text{int } D^2) \cup_{\partial\gamma \times S^1} (\gamma \times S^1),$$

which has boundary δ . Note that $\Gamma_i(\Theta)$ is disjoint from $f(S_+^2)$. Form a loop β on $\mathbb{R}^3[1] \cap \Theta$ by connecting the endpoints of $\gamma \times \{1\}$ by an arc on $\hat{\Delta} \setminus (\partial\gamma \times \text{int } D^2)$.

Since $\pi_1(X_-) \cong \mathbb{Z}$ and a loop of the form $\Gamma_i(\{b\} \times S^1)$ ($b \in \text{int } \gamma$) is meridinal to $f(S_-^2)$, by replacing $\gamma \times \{1\} \subset \beta$ by its band sum with oriented copies of $\{b\} \times S^1$ if necessary (see Figure 3(b)) we may assume that $\beta_i = \Gamma_i(\beta)$ is a simple circle bounding an immersed, self-transverse 2-disk D_i in X_- that is transverse to $f(S_+^2)$.

Now, as $f(S_+^2)$ misses $\Gamma_i(N^+)$, the loop β_i is freely homotopic in $S^4 \setminus f(S_+^2)$ to α_i . Consequently,

$$|f(S_+^2) \cdot D_i| = |\text{lk}(f(S_+^2), \alpha_i)| \quad (2)$$

as non-negative integers. Let \widehat{A}_i be the immersed, self-transverse 2-disk in X_- obtained by performing the construction of (3.1.1) with the (embedded) punctured torus $\Gamma_i(\Theta) \subset X_- \setminus f(S_+^2)$, $\beta_i \subset \Gamma_i(\Theta)$, D_i and some choice of normal vector field to β_i on $\Gamma_i(\Theta)$. Then, since a loop of the form $\Gamma_i(\{b\} \times S^1)$ ($b \in \text{int } \gamma$) is dual to β_i on $\Gamma_i(\Theta)$ and hence represents a generator (s or s^{-1}) of $\pi_1(X_-)$, by Equation (2) and Lemma 6 we have (after orienting \widehat{A}_i and connecting it to the basepoint of X_-)

$$\lambda(f(S_+^2), \widehat{A}_i) = (1 - s)\widehat{q}_i(s) \quad (3)$$

for some integer Laurent polynomial $\widehat{q}_i \in \mathbb{Z}[s, s^{-1}]$ such that $\widehat{q}_i(1) = \text{lk}(f(S_+^2), \alpha_i)$. Moreover, if $|f(S_+^2) \cap D_j| = 1$ for some j then (since we are free to choose the orientation and whisker of \widehat{A}_j) we may take $\widehat{q}_j = 1$.

Now, for each i , \widehat{A}_i is bound by the circle $\Gamma_i(\delta)$ on the (embedded) linking torus $T_i \subset X_- \setminus f(S_+^2)$. Perform the construction of (3.1.1) with T_i , $\Gamma_i(\delta)$, \widehat{A}_i and some choice of normal vector field to $\Gamma_i(\delta)$ on T_i . Then, since δ has a dual curve on T that is meridinal to U , by Equation (3) and Lemma 6 we have (after orienting A_i and connecting it to the basepoint of X_-)

$$\lambda(f(S_+^2), A_i) = (1 - s)^2 q_i(s)$$

for some $q_i \in \mathbb{Z}[s, s^{-1}]$ such that $q_i(1) = \text{lk}(f(S_+^2), \alpha_i)$. As above, if $|f(S_+^2) \cap D_j| = 1$ for some j then we may take $q_j = 1$.

By construction, A_i is homologous to T_i for each i , so by Lemma 8 the immersed 2-spheres $\{A_i\}_{i=1}^d$ represent a $\mathbb{Z}[s, s^{-1}]$ -basis for $\pi_2(X_-)$. \square

3.3 The invariant τ applied to link maps

In [9], the authors define a homotopy invariant τ which takes as input a map $k : S^2 \rightarrow Y^4$ with vanishing Wall self-intersection $\mu(k)$ and gives output in a quotient $\Pi(Y, k)$ of the group ring $\mathbb{Z}[\pi_1(Y) \times \pi_1(Y)]$ modulo certain relations. The relations are additively generated by the equations

$$(a, b) = -(b, a) \quad (\mathcal{R}_1)$$

$$(a, b) = -(a^{-1}, ba^{-1}) \quad (\mathcal{R}_2)$$

$$(a, 1) = (a, a) \quad (\mathcal{R}_3)$$

$$(a, \lambda(k(S^2), A)) = (a, \omega_2(A) \cdot 1) \quad (\mathcal{R}_4)$$

where $a, b \in \pi_1(Y)$, A represents an immersed S^2 or \mathbb{RP}^2 in Y (in the latter case, the group element a is the image of the nontrivial element in $\pi_1(\mathbb{RP}^2)$).

Let f be a good link map with $\sigma_+(f) = 0$ (from which it follows that $\mu(f_+) = 0$). For an integer k , let \bar{k} denote its image in \mathbb{Z}_2 . Letting ρ denote the mod 2 Hurewicz map $\pi_1(X_-) \rightarrow H_1(X_-; \mathbb{Z}_2) = \mathbb{Z}_2$, define a ring homomorphism $\varphi_f : \Pi(X_-, f_+) \rightarrow \mathbb{Z}_2\langle t : t^2 = 1 \rangle$ by

$$(a, b) \mapsto t^{\overline{\rho(a) + \rho(a)\rho(b) + \rho(b)}},$$

and extending linearly mod 2. We now prove a stronger form of Theorem 2.

Lemma 10. *Let f be a link map with $\sigma_+(f) = 0$. After a certain link homotopy of f we have that φ_f is an isomorphism and takes $\tau(f_+)$ to $(1 + t)\omega_+(f)$.*

Proof. By Lemma 7 we may assume f is a good link map (and so $\mu(f_+) = \sigma_+(f) = 0$) such that $f(S_-^2)$ is unknotted. We may perform a finger move of $f(S_-^2)$ along an chord attached in the complement of and meridinal to $f(S_-^2)$ so that a slice in $\mathbb{R}^3[t_0]$ (for some t_0) of the result is illustrated in Figure 4. This produces a pair of oppositely-signed double points $\{p^+, p^-\}$ on $f(S_-^2)$ such that (in particular) p^+ has an accessory circle bounding an obvious embedded 2-disk in $\mathbb{R}^3[t_0]$ that intersects $f(S_+^2)$ exactly once. Note that $f(S_-^2)$ is still unknotted (by [3, Lemma 1]) and, in particular, its complement in S^4 still has abelian fundamental group.

Now, fixing f_- and X_- , by Lemma 9 we may thus identify $\pi_1(X_-)$ with $\mathbb{Z}\langle s \rangle$ and $\pi_2(X_-)$ with $(\bigoplus_{i=1}^d \mathbb{Z})[s, s^{-1}]$, for some $d \geq 0$, such that there is an immersed, whiskered 2-sphere A_0 in X_- with the property that $\lambda(f(S_+^2), A_0) = (1 - s)^2$. Moreover, for any whiskered, immersed 2-sphere A in X_- we have $\lambda(f(S_+^2), A) = (1 - s)^2 q_A(s)$ for some integer Laurent polynomial $q_A \in \mathbb{Z}[s, s^{-1}]$.

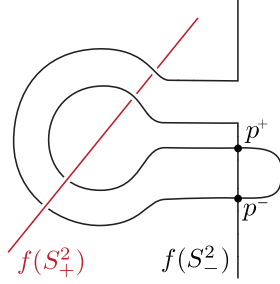


Figure 4

Therefore, if we identify $\mathbb{Z}[\pi_1(X_-) \times \pi_1(X_-)]$ with $\mathbb{Z}[s^{\pm 1}, t^{\pm 1}]$ via $(s^n, s^m) = s^n t^m$ (for $n, m \in \mathbb{Z}$), by Lemma 8 the ring $\Pi(X_-, f_+)$ is the quotient of the group

ring $\mathbb{Z}[s^{\pm 1}, t^{\pm 1}]$ modulo the relations generated additively by the equations:

$$s^n t^n - s^n = 0 \quad (\mathcal{T}_1)$$

$$s^n t^m + s^{-n} t^{m-n} = 0 \quad (\mathcal{T}_2)$$

$$s^n t^m + s^m t^n = 0 \quad (\mathcal{T}_3)$$

$$s^n t^m (1-t)^2 = 0 \quad (\mathcal{T}_4)$$

where $n, m \in \mathbb{Z}$. Note that in reformulating Relation (\mathcal{R}_4) to obtain Relation (\mathcal{T}_4) we have used the action of $\pi_1(X_-) = \mathbb{Z}\langle s \rangle$ on $\pi_2(X_-)$.

Let \equiv denote equivalence in $\Pi(X_-, f_+)$. Clearly φ_f is surjective; to show injectivity we first show that for any integers n, m , one has

$$s^n t^m \equiv t^{\overline{n+nm+m}}. \quad (4)$$

By Relations (\mathcal{T}_2) and (\mathcal{T}_3) we have $2t^n \equiv 0$ and hence $2s^n \equiv -2t^n \equiv 0$ for each $n \in \mathbb{Z}$. Then Relation (\mathcal{T}_4) implies that $t^{m+2} \equiv t^m$ for any integer m , and it follows by an induction argument that

$$t^m \equiv t^{\overline{m}}. \quad (5)$$

Now, $s \equiv -t \equiv t$ and $st \equiv t$ by Relations (\mathcal{T}_1) – (\mathcal{T}_3) . Combining these equivalences with the consequence of Relation (\mathcal{T}_4) that $st^{m+2} \equiv 2st^{m+1} - st^m$, an induction gives

$$st^m \equiv t \quad (6)$$

for any integer m .

Finally, fix $n_0 \in \mathbb{Z}$. By Relation (\mathcal{T}_3) and Equivalences (5) and (6), we have $s^{n_0} \equiv -t^{n_0} \equiv t^{\overline{n_0}}$ and $s^{n_0} t \equiv -st^{n_0} \equiv -t \equiv t$. Suppose now that for some $k \geq 1$ Equivalence (4) holds for $n = n_0$ and any $m \in \{0, 1, \dots, k\}$. Then Relation (\mathcal{T}_4) implies that

$$\begin{aligned} s^{n_0} t^{k+1} &\equiv 2s^{n_0} t^k - s^{n_0} t^{k-1} \\ &\equiv 2t^{\overline{n_0+n_0 k+k}} - t^{\overline{n_0+n_0(k-1)+(k-1)}} \\ &\equiv t^{\overline{n_0+n_0(k+1)+(k+1)}}. \end{aligned}$$

On the other hand, suppose that for some $k \leq 0$ Equivalence (4) holds for $n = n_0$ and any $m \in \{k, k+1, \dots, 0, 1\}$; then

$$\begin{aligned} s^{n_0} t^{k-1} &\equiv 2s^{n_0} t^k - s^{n_0} t^{k+1} \\ &\equiv 2t^{\overline{n_0+n_0 k+k}} - t^{\overline{n_0+n_0(k+1)+(k+1)}} \\ &\equiv t^{\overline{n_0+n_0(k-1)+(k-1)}}. \end{aligned}$$

Thus, by induction Equivalence (4) holds for $n = n_0$ and any integer m . But $n_0 \in \mathbb{Z}$ was arbitrary, so the equivalence holds for all integers n, m . As 2 \equiv

$0 \equiv 2t$, we deduce that $\Pi(X_-, f_+)$ is the group ring $\mathbb{Z}_2\langle t : t^2 = 1 \rangle$ and φ_f is injective.

Turning to the second part of the lemma, we refer the reader to [6] and [9] for detailed descriptions of the ω and τ invariants, respectively, and to [2] for background on framed Whitney disks. We make only a few summarizing remarks.

Since $\sigma_+(f) = 0$, $\pi_1(X_-) = \mathbb{Z}\langle s \rangle$, and f_+ is self-transverse with vanishing signed sum of its double points, the double points of $f(S_+^2)$ may be decomposed into *canceling* pairs $\{p_i^+, p_i^-\}_{i=1}^k$ in the following sense. For each $1 \leq i \leq k$, one has $\text{sign}(p_i^+) = -\text{sign}(p_i^-)$ and the preimages of p_i^\pm in S_+^2 may be labeled $\{x_i^\pm, y_i^\pm\}$ so that if γ_i is an arc on S_+^2 connecting x_i^+ to x_i^- (and missing all other double point preimages) and γ'_i is an arc on S_+^2 connecting y_i^+ to y_i^- (and missing γ_i and all other double point preimages), then the loop $f(\gamma_i) \cup f(\gamma'_i) \subset f(S_+^2)$ is nullhomotopic in X_- . The arcs $\{\gamma_i, \gamma'_i\}_{i=1}^k$ may be chosen so that the resulting *Whitney circles* $\{f(\gamma_i \cup \gamma'_i)\}_{i=1}^k$ are mutually disjoint, simple circles in X_- such that each bounds an immersed, framed Whitney disk W_i in X_- whose interior is transverse to $f(S_+^2)$. Let α_i^\pm be an arc on S_+^2 connecting x_i^\pm to y_i^\pm , and let n_i^\pm denote the integer $\text{lk}(f(S_+^2), f(\alpha_i^\pm))$; then $n_i^+ = -n_i^-$. (In [6] the non-negative integer $|n_i^+|$ is called the *n-multiplicity* for the pair $\{p_i^+, p_i^-\}$, and in [9] the $\pi_1(X_-)$ -element $s^{n_i^+}$ is called the *primary group element* for W_i .) Note that $\rho(s^{n_i^+})$ is the mod 2 image of n_i^+ .

Let $i \in \{1, 2, \dots, k\}$ and suppose $x \in f(S_+^2) \cap \text{int } W_i$. A loop that first goes along $f(S_+^2)$ from its basepoint to x , then along W_i to $f(\gamma'_i) \subset \partial W_i$, then back along $f(S_+^2)$ to the basepoint of $f(S_+^2)$, determines a $\pi_1(X_-)$ -element s^{m_x} (called the *secondary group element* associated to x in [9]; the non-negative integer $|m_x|$ is called the *m-multiplicity* of x in [6]). Associate to x a sign by orienting W_i using the following convention: orient ∂W_i from p_i^- to p_i^+ along the $f(\gamma'_i)$, then back to p_i^- along $f(\gamma_i)$; the positive tangent to ∂W_i together with an outward-pointing second vector then orient W_i . Let

$$J_x^i = \overline{n_i^+ + n_i^+ m_x + m_x} \in \mathbb{Z}_2$$

and

$$I_x^i = \text{sign}(x) s^{n_i^+} t^{m_x} \in \mathbb{Z}[s^{\pm 1}, t^{\pm 1}].$$

Then Li's \mathbb{Z}_2 -valued ω_+ -invariant applied to f is defined by

$$\omega_+(f) = \sum_{i=1}^k \sum_{x \in f(S_+^2) \cap \text{int } W_i} J_x^i \pmod{2};$$

while, in this special case, the Schneiderman-Teichner invariant τ applied to f_+ is given by the $\mathbb{Z}[s^{\pm 1}, t^{\pm 1}]$ -sum

$$\tau(f_+) = \sum_{i=1}^k \sum_{x \in f(S_+^2) \cap \text{int } W_i} I_x^i$$

evaluated in the quotient $\Pi(X_-, f_+)$.

Now, $\varphi_f(I_x^i) = t^{J_x^i}$, and consequently

$$\varphi_f(\tau(f_+)) = \sum_{i=1}^k \sum_{x \in f(S_+^2) \cap \text{int } W_i} t^{J_x^i}.$$

But $\varphi(\tau(f_+)) \in \{0, 1, t, 1+t\}$ must map forward to 0 under the homomorphism $\Pi(X_-, f_+) \rightarrow \Pi(S^4, f_+) = \mathbb{Z}_2$ induced by the inclusion $X_- \subset S^4$ and given by sending $s, t \mapsto 1$. Thus

$$\begin{aligned} \varphi_f(\tau(f_+)) &\equiv \sum_{i=1}^k \sum_{x \in f(S_+^2) \cap \text{int } W_i} J_x^i \cdot (1+t) \pmod{2} \\ &= (1+t)\omega_+(f). \end{aligned} \quad \square$$

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